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Results in Mathematics

The Surface Group Conjecture: Cyclically Pinched and Conjugacy Pinched One-Relator Groups

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In memoriam Colin Maclachlan.

Abstract. The general **surface group conjecture** asks whether a one-relator group where every subgroup of finite index is again one-relator and every subgroup of infinite index is free (property IF) is a surface group. We resolve several related conjectures given in Fine et al. (Sci Math A 1:1–15, 2008). First we obtain the Surface Group Conjecture B for cyclically pinched and conjugacy pinched one-relator groups. That is: if G is a cyclically pinched one-relator group or conjugacy pinched one-relator group satisfying property IF then G is free, a surface group or a solvable Baumslag–Solitar Group. Further combining results in Fine et al. (Sci Math A 1:1–15, 2008) on Property IF with a theorem of Wilton (Geom Topol, 2012) and results of Stallings (Ann Math 2(88):312–334, 1968) and Kharlampovich and Myasnikov (Trans Am Math Soc 350(2):571–613, 1998) we show that Surface Group Conjecture C proposed in Fine et al. (Sci Math A 1:1–15, 2008) is true, namely: If G is a finitely generated nonfree freely indecomposable fully residually free group with property IF, then G is a surface group.

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1. Introduction

The **surface group conjecture** as originally proposed in the Kourovka notebook [21] by Melnikov was the following problem.

Surface Group Conjecture. *Suppose that G is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then G is a surface group.*

In this form the conjecture is false. Recall that the Baumslag–Solitar groups $BS(m,n)$ are the groups

$$BS(m,n) = \left\langle a, b; a^{-1}b^m a = b^n \right\rangle .$$

If $|m| = |n|$ or either $|m| = 1$ or $|n| = 1$ these groups are residually finite. In all other cases these groups are not residually finite. Further $BS(m,n)$ is Hopfian if and only if $m = \pm 1$ or $n = \pm 1$ or m, n have the same set of primes ([5]). If either $|m| = 1$ or $|n| = 1$ every subgroup of finite index is again a Baumslag–Solitar group and therefore a one-relator group, and every subgroup of infinite index is infinite cyclic and therefore a free group of rank one ([10]). It follows that besides the surface groups, the groups $BS(1,m)$, also satisfy Melnikov's question. We then have the following modified conjecture.

Surface Group Conjecture A. *Suppose that G is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then G is either a surface group or a Baumslag–Solitar group $BS(1,m)$ for some integer m.*

We note that the groups $BS(1,1)$ and $BS(1,-1)$ are surface groups. In surface groups, subgroups of infinite index must be free groups. To avoid the Baumslag–Solitar groups, $BS(1,m), |m| \geq 2$, Surface Group Conjecture A, was modified to:

Surface Group Conjecture B. *Suppose that G is a non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group, there exists a noncyclic subgroup of infinite index and every subgroup of infinite index is a free group. Then G is a surface group of genus $g \geq 2$.*

In [6] the surface group conjecture was considered for fully residually free groups using the JSJ decomposition. A group G has **Property IF** if every subgroup of infinite index in G is free. In [6] it was proved that a fully residually free group satisfying property IF is either a cyclically pinched one-relator group or a conjugacy pinched one-relator group. This led to the following:

Surface Group Conjecture C. *Suppose that G is a finitely generated nonfree freely indecomposable fully residually free group with property IF. Then G is a surface group.*

In Theorem 3.2, we settle Surface Group Conjecture C by combining results in [6] with a recent theorem of Wilton [26] and results of Kharlampovich and Myasnikov [19] and Stallings [23]. This result also appears in Wilton [26] for one-ended limit groups.

In Theorem 3.1 we then improve on a result of Wilton (see Theorem 2.4) by dropping the conditions of one-ended hyperbolic from the hypothesis. Our main result concerns cyclically pinched and conjugacy pinched one-relator groups (see Theorem 3.1).

2. Background Material and Necessary Results

Let G be the fundamental group of a compact surface of genus g. Then G has a one-relator presentation

$$\langle a_1, b_1, \ldots, a_g, b_g; [a_1, b_1], \ldots, [a_g, b_g]\rangle$$

in the orientable case and

$$\left\langle a_1, \ldots, a_g; a_1^2, \ldots, a_g^2\right\rangle$$

in the non-orientable case. From covering space theory it follows that any subgroup of finite index is again a surface group of the same or higher genus while any subgroup of infinite index must be a free group. These results, although known since the early 1900s, were proved purely algebraically using Reidemeister-Schreier rewriting by Hoare et al. [11,12]. It is well known (see [7]) that an orientable surface group can be faithfully represented as a discrete subgroup of $PSL_2(\mathbb{C})$ and hence each such group is linear. It follows that surface groups are residually finite. Baumslag [2] showed that any orientable surface group of genus ≥ 2 must be residually free and 2-free from which it can be deduced using results of Remeslennikov [22] and Gaglione and Spellman [9] that they are fully residually free (see Sect. 2). The article [1] surveys most of the properties of surface groups and shows how they are the primary motivating examples for much of combinatorial group theory.

A **cyclically pinched one-relator group** is a one-relator group of the following form

$$G = \langle a_1, \ldots, a_p, a_{p+1}, \ldots, a_n; U = V\rangle$$

where $1 \neq U = U(a_1, \ldots, a_p)$ is a cyclically reduced word in the free group F_1 on $a_1, \ldots, a_p$ and $1 \neq V = V(a_{p+1}, \ldots, a_n)$ is a cyclically reduced in the free group F_2 on $a_{p+1}, \ldots, a_n$.

Clearly such a group is the free product of the free groups on $a_1, \ldots, a_p$ and $a_{p+1}, \ldots, a_n$, respectively, amalgamated over the cyclic subgroups generated by U and V. From the standard one-relator presentation for an orientable surface group of genus $g \geq 2$ it follows that they are cyclically pinched one-relator groups. There is a similar decompositon in the nonorientable case.

The HNN analogs of cyclically pinched one-relator groups are called **conjugacy pinched one-relator groups** and are also motivated by the structure of orientable surface groups. In particular suppose

$$S_g = \langle a_1, b_1, \ldots, a_g, b_g; [a_1, b_1], \ldots, [a_g, b_g] = 1\rangle .$$

If $b_g = t$ then S_g is an HNN group of the form

$$S_g = \left\langle a_1, b_1, \ldots, a_g, t; tUt^{-1} = V \right\rangle$$

where $U = a_g$ and $V = [a_1, b_1], \ldots, [a_{g-1}, b_{g-1}]a_g$. Generalizing this we say that a **conjugacy pinched one-relator group** is a one-relator group of the form

$$G = \left\langle a_1, \ldots, a_n, t; tUt^{-1} = V \right\rangle$$

where $1 \neq U = U(a_1, \ldots, a_n)$ and $1 \neq V = V(a_1, \ldots, a_n)$ are cyclically reduced in the free group F on $a_1, \ldots, a_n$.

The one-relator presentation for a surface group allows for a decomposition as a cyclically pinched one-relator group in both the orientable and non-orientable cases and as a conjugacy pinched one relator group in the orientable case (see [8]). In general cyclically pinched one-relator groups and conjugacy pinched one-relator groups have been shown to be extremely similar to surface groups. We refer to [7] or [8], where a great deal more information about both cyclically pinched one-relator groups and conjugacy pinched one-relator groups is available.

In [6] several results were proved about the surface group conjectures. Recall that a group G is *fully residually free* if given finitely many nontrivial elements $g_1, \ldots, g_n$ in G there is a homomorphism $\phi : G \to F$, where F is a free group, such that $\phi(g_i) \neq 1$ for all $i = 1, \ldots, n$. Fully residually free groups have played a crucial role in the study of equations and first order formulas over free groups and in particular the solution of the Tarski problem (see [16–20, 24, 25]). Finitely generated fully residually free groups are also known as **limit groups**. In this guise they were studied by Sela (see [4, 24]) in terms of studying homomorphisms of general groups into free groups.

In [6] several results concerning the Surface Group Conjectures were obtained:

Theorem 2.1. (Theorem 3.1, [6]) *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either a free group or a cyclically pinched one relator group or a conjugacy pinched one relator group.*

Corollary 2.2. (Corollary 3.2, [6]) *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either free or every subgroup of finite index is freely indecomposable and hence a one-relator group.*

Furthermore we have:

Theorem 2.3. (Theorem 3.2, [6]) *Let G be a finitely generated fully residually free group with property IF. Then G is either hyperbolic or free abelian of rank 2.*

These results depend on the fact that the fully residually free groups have JSJ-decompositions. Roughly a JSJ-decomposition of a group G is a splitting of G as a graph of groups with abelian edges which is canonical in that it encodes all other such abelian splittings. If each edge is cyclic it is called a *cyclic JSJ-decomposition* (see [6]).

Being fully residually free provides a graph of groups decomposition. Then property IF will imply the finite index property, as follows:

Theorem 2.4. (Theorem 3.3, [6]) *Let G be a nonfree cyclically pinched or conjugacy pinched one-relator group with property IF. Then each subgroup of finite index is again a cyclically pinched or conjugacy pinched one-relator group.*

The proof of Theorem 2.4 used the subgroup theorems for free products with amalgamation and HNN groups as described by Karrass and Solitar [14,15].

Our proof of the Surface Group Conjecture C in Theorem 3.2 combines the results in [6] with the following statement, which is a rewording of a recent result of Wilton [26].

Theorem 2.5. (see Corollary 4, [26]) *Let G be a hyperbolic one-ended cyclically pinched one-relator group or a hyperbolic one-ended conjugacy pinched one-relator group. Then either G is a surface group, or G has a finitely generated non-free subgroup of infinite index.*

3. Main Results

Our main result shows that Surface Group Conjecture B is true if G is a cyclically pinched or conjugacy pinched one-relator.

Theorem 3.1. (1) *Let G be a cyclically pinched one-relator group satisfying property IF. Then G is a free group or a surface group.*

(2) *Let G be a conjugacy pinched one-relator group satisfying property IF. Then G is a free group, a surface group or a solvable Baumslag–Solitar group.*

Recall that a Baumslag–Solitar group $BS(m, n)$ is solvable if and only if $|m| = 1$ or $|n| = 1$.

Before proving Theorem 3.1, we settle the Surface Group Conjecture C in the following theorem. This result appears in (Corollary 5, [26]) for one-ended limit groups.

Theorem 3.2. *Suppose that G is a finitely generated nonfree freely indecomposable fully residually free group with property IF. Then G is a surface group. That is, Surface Group Conjecture C is true.*

Proof. Suppose that G is a finitely generated freely indecomposable fully residually free group with property IF. If G is free abelian we are done since the free abelian group of rank 2 is a surface group, and of higher rank does not posses IF. If it is not free abelian then from Theorem 1.2 from [6] it follows that G is hyperbolic.

Since G is assumed to be a finitely generated freely indecomposable fully residually group and satisfies Property IF, from Theorem 2.1 in [6] it follows that G must be either a cyclically pinched one relator group or a conjugacy pinched one relator group.

To apply Wilton's Theorem we show that it must be one-ended. Let $e(H)$ denote the number of ends of a group H. From a theorem of Stallings [23] $e(H) = 0$ if and only if H is finite and $e(H) > 1$ if and only if H has a non-trivial decomposition as a free product with amalgamation with finite amalgamated subgroup or a non-trivial decomposition as a HNN-group with finite associated subgroup.

Let G be the group as in the statement of the theorem. Certainly G is not finite so $e(G) > 0$. Since G satisfies Property IF it follows that if $e(G) > 1$ then the amalgamated subgroup (or the associated subgroup) in G has to be trivial. That implies that if our G is not one-ended then there is a free infinite cyclic factor and G is free or has a non-free subgroup of infinite index. Since G is freely indecomposable it follows that it must be one-ended and Wilton's result applies.

Therefore G is either a surface group or has a nonfree subgroup of infinite index. Again from property IF G must be a surface group settling Surface Group Conjecture C. $\square$

We now give the proof for cyclically pinched and conjugacy pinched one-relator groups.

Proof. (Theorem 3.1) (1) Let G be a cyclically pinched one-relator group amalgamated via $U = V$ and suppose that G is nonfree. Suppose that not both U and V are proper powers. Then by results of Juhasz and Rosenberger [13], Bestvina and Feighn [3] and Kharlampovich and Myasnikov [19], G must be hyperbolic. Since G satisfies Property IF, as in the proof of Theorem 3.2, G must be one-ended and hence Wilton's theorem applies to give that G must be a surface group.

Suppose now that both U and V are proper powers. Let $U = g^n, n > 1$ and $V = h^m, m > 1$. If $G = \langle g, h : g^2 = h^2 \rangle$ then G is a nonhyperbolic, nonorientable surface group of genus 2.

Now assume that G is not isomorphic to a group $\langle a, b; a^2 = b^2 \rangle$. Then consider the subgroup $H = \langle g^n, gh \rangle$. H is free abelian of rank 2 and further H has infinite index in G. To see this introduce the relations $g^n = h^m = 1$. Then the image of G is a nontrivial free product, not isomorphic to the infinite dihedral group, and the image of H is infinite cyclic. However G is assumed to have Property IF and hence this case is impossible.

Thus all cyclically pinched one-relator groups with Property IF must be either free or a surface groups.

(2) Now let G be a conjugacy pinched one-relator group satisfying Property IF and assume G is nonfree. Suppose first that U and V are not both proper powers. Assume that U and V are conjugately separated in F, where F is the group generated by $a_1, \ldots, a_n$. This means that $\langle U\rangle \cap x\langle V\rangle x^{-1}$ is finite for all $x \in F$. Since G is torsion-free this intersection must be trivial. By a result of Kharlampovich and Myasnikov [19], G is then hyperbolic. Since G satisfies Property IF then, as in the proof of Theorem 3.2, G is one-ended and hence Wilton's theorem applies to give that G is a surface group.

Thus $\langle U\rangle \cap x\langle V\rangle x^{-1}$ is infinite, in fact infinite cyclic, for some $x \in F$. After a suitable conjugation and a possible interchange of U and V we may assume that U is not a proper power in F and $V = U^k$ for some $k \neq 0$. Let K be the subgroup generated by U and t. By normal form arguments K has a presentation $K = \langle U, t : tUt^{-1} = U^k\rangle$ (see [8]). If F is free of rank > 1, that is if $n > 1$, then K is a nonfree subgroup of infinite index in G. This can be seen as follows. In G introduce the relations $t = U = 1$. Then the image of G is a one-relator group $\langle a_1, \ldots, a_n; U = 1\rangle$ which is infinite by the Freiheitssatz. Therefore this case cannot occur since G satisfies Property IF.

Now let F be cyclic and let $a_1 = a$, so $G = \langle a, t; tat^{-1} = a^p\rangle$ with $p = \pm k$. Then G is a solvable Baumslag–Solitar group.

Finally let both U and V be proper powers so suppose that $U = g^n, n > 1$ and $V = h^m, m > 1$. By normal form arguments the subgroup N generated by $w = tgt^{-1}$ and h has a presentation $N = \langle w, h; w^n = h^m\rangle$ (see [8]). Note that the subgroup H generated by w^n and wh is free abelian of rank 2 and has infinite index in G. Since G is assumed to have Property IF this does not occur.

Altogether, if G is a conjugacy pinched one-relator group with Property IF then G must be either free, a surface group or a solvable Baumslag–Solitar group. $\square$

4. Some Observations on the General Conjecture

Here we make some straightforward observations based on the proofs of Theorems 3.1 and 3.2 that might have a bearing on the general Surface Group Conjecture. From the proofs in [6] and the proofs of Theorems 3.1 and 3.2 we have the following.

Lemma 4.1. *Let G be nonfree and have a graph of groups decomposition with Property IF. Then each factor must be free and G is one-ended.*

Lemma 4.2. *Let G be nonfree and have a graph of groups decomposition with cyclic edge groups and with Property IF. Then G is a cyclically pinched or conjugacy pinched one-relator group, G is one-ended and hence either a surface group or a solvable Baumslag–Solitar group.*

We would like to note how strong Property IF is. A rewording of Lemma 4.2 implies that if G is not a free, not a surface, and not a Baumslag–Solitar group, then it cannot have a graph of groups decomposition with cyclic edge groups.

Now we consider noncyclic one-relator groups. Certainly if G has property IF it must be torsion-free. Let G be a torsion-free one-relator group with property IF. Using the standard Magnus breakdown G can be taken to be an HNN group. Hence by Property IF the base F is a free group and as in the proofs in Sect. 3 it must be one-ended. Altogether then if G is a torsion-free finitely generated one-relator group with property IF then G has a presentation as an HNN extension with base group F, a free group. Then by rewriting we can get a presentation of the form

$$G = \langle F, t; tU_1t^{-1} = U_2, tU_2t^{-1} = U_3, \ldots, tU_{k-1}t^{-1} = U_k \rangle$$

for some free group words $U_1, U_2, \ldots, U_k$. If $k = 2$ then it is a conjugacy pinched one-relator group and hence G is either free, a surface group or a solvable Baumslag–Solitar group. The general surface group conjecture would then be true if the following one is true.

Conjecture. *Suppose that*

$$G = \langle F, t; tU_1t^{-1} = U_2, tU_2t^{-1} = U_3, \ldots, tU_{k-1}t^{-1} = U_k \rangle$$

with F a finitely generated nonabelian free group and $U_1, U_2, \ldots, U_k$ some nontrivial free group words. If $k > 2$ there must be a finite index subgroup that is not a one-relator group.

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References

[1] Ackermann, P., Fine, B., Rosenberger, G.: On Surface Groups: Motivating Examples in Combinatorial Group Theory, Groups St. Andrews 2005, pp. 96–129. Cambridge University Press, Cambridge (2007)

[2] Baumslag, G. : On generalized free products. Math. Z. **78**, 423–438 (1962)

[3] Bestvina, M., Feighn, M.: A combination theorem for negatively curved groups. J. Diff. Geom. **35**, 85–101 (1992)

[4] Bestvina, M., Feighn, M.: Notes on Sela's Work: Limit Groups and Makanin–Razborov Diagrams, preprint

[5] Collins, D., Levin, F.: Automorphisms and Hopficity of certain Baumslag–Solitar groups. Arch. Math. **40**, 385–400 (1983)

[6] Fine, B., Kharlampovich, O., Myasnikov, A., Remeslennikov, V., Rosenberger, G.: On the Surface Group Conjecture. Sci. Math A **1**, 1–15 (2008)

[7] Fine, B., Rosenberger, G.: Algebraic Generalizations of Discrete Groups. Marcel-Dekker, New York (1999)

[8] Fine, B., Rosenberger, G., Stille, M.: Conjugacy pinched and cyclically pinched one-relator groups. Rev. Math. Madr. **10**, 207–227 (1997)

[9] Gaglione, A., Spellman, D.: Some model theory of free groups and free algebras. Houst. J. Math. **19**, 327–356 (1993)

[10] Gildenhuys, D., Kharlampovich, O., Myasnikov, A.: CSA groups and separated free constructions. Trans. Am. Math. Soc. **350**, 571–613 (1998)

[11] Hoare, A., Karrass, A., Solitar, D.: Subgroups of finite index of Fuchsian groups. Math. Z. **120**, 289–298 (1971)

[12] Hoare, A., Karrass, A., Solitar, D.: Subgroups of infinite index in Fuchsian groups. Math. Z. **125**, 59–69 (1972)

[13] Juhasz, A., Rosenberger, G.: On the combinatorial curvature of groups of F-type and other one-relator products of cyclics. Cont. Math. **169**, 373–384 (1994)

[14] Karrass, A., Solitar, D.: The subgroups of a free product of two groups with an amalgamated subgroup. Trans. Am. Math. Soc. **150**, 227–255 (1970)

[15] Karrass, A., Solitar, D.: Subgroups of HNN groups and groups with one defining relation. Can. J. Math. Soc. **23**, 627–643 (1971)

[16] Kharlamapovich, O., Myasnikov, A.: Irreducible affine varieties over a free group: I. Irreducibility of quadratic equations and Nullstellensatz. J. Algebra **200**, 472–516 (1998)

[17] Kharlamapovich, O., Myasnikov, A.: Irreducible affine varieties over a free group: II. Systems in triangular quasi-quadratic form and a description of residually free groups. J. Algebra **200**, 517–569 (1998)

[18] Kharlamapovich, O., Myasnikov, A.: Description of fully residually free groups and Irreducible affine varieties over free groups, Summer school in Group Theory in Banff, 1996, CRM Proceedings and Lecture notes, 17, pp. 71–81 (1999)

[19] Kharlamapovich, O., Myasnikov, A.: Hyperbolic groups and free constructions. Trans. Am. Math. Soc. **350**(2), 571–613 (1998)

[20] Kharlamapovich, O., Myasnikov, A.: Description of fully residually free groups and irreducible affine varieties over a free group, CRM Proceeding and Lecture Notes: Summer School in Group Theory in Banff 1996, 17, pp. 71–80, 1998

[21] Merzlyakov, Y.I.: Kourovka Notebook—Unsolved Problems in Group Theory

[22] V.N. Remeslennikov: ∃-free groups. Sib. Mat. J. **30**, 998–1001 (1989)

[23] Stallings, J.: On torsion-free groups with infinitely many ends. Ann. Math. **2**(88), 312–334 (1968)

[24] Sela, Z.: Diophantine geometry over groups I: Makanin-Razborov diagrams. Publ. Math. IHES **93**, 31–105 (2001)

[25] Sela, Z.: Diophantine geometry over groups V: Quantifier elimination. Isr. J. Math. **150**, 1–97 (2005)

[26] Wilton, H.: One ended subgroups of graphs of free groups with cyclic edge groups. Geom. Topol. **16**, 665–683 (2012)

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